Dimensional Analysis and Similitude

Modeling at aeromechanical experiment. Similarity criterion. (Re, Pe, Fr, Nu, Pr, Sh and etc.)

Main Topics

- Nature of Dimensional Analysis
- Buckingham Pi Theorem
- Significant Dimensionless Groups in Fluid Mechanics
- Flow Similarity and Model Studies

Objectives

1. Understand dimensions, units, and dimensional homogeneity
2. Understand benefits of dimensional analysis
3. Know how to use the method of repeating variables
4. Understand the concept of similarity and how to apply it to experimental modeling

Dimensions and Units

- Review
  - Dimension: Measure of a physical quantity, e.g., length, time, mass
  - Units: Assignment of a number to a dimension, e.g., (m), (sec), (kg)
  - 7 Primary Dimensions:

1.	**Mass**	**m**	**(kg)**
2.	**Length**	**L**	**(m)**
3.	**Time**	**t**	**(sec)**
4.	**Temperature**	**T**	**(K)**
5.	**Current**	**I**	**(A)**
6.	**Amount of Light**	**C**	**(cd)**
7.	**Amount of matter**	**N**	**(mol)**

Dimensions and Units

- All non-primary dimensions can be formed by a combination of the 7 primary dimensions
- Examples
  - **{Velocity} m/sec = {Length/Time} = {L/t}**
  - **{Force} N = {Mass Length/Time} = {mL/t^2}**

Dimensional Homogeneity

- Every additive term in an equation must have the same dimensions
- Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

  - $\{p\} = \{force/area\} = \{mass \times length/time \times 1/length^2\} = \{m/(t^2L)\}$
  - $\{1/2\rho V^2\} = \{mass/length^3 \times (length/time)^2\} = \{m/(t^2L)\}$
  - $\{\rho g z\} = \{mass/length^3 \times length/time^2 \times length\} = \{m/(t^2L)\}$

Nature of Dimensional Analysis

Example: Drag on a Sphere

$$F = f(D,\ V,\ \rho,\ \mu)$$

- ✓ **Drag depends on FOUR parameters: sphere size (*D*); speed (*V*); fluid density (ρ); fluid viscosity (μ)**
- ✓ **Difficult to know how to set up experiments to determine dependencies**
- ✓ **Difficult to know how to present results (four graphs?)**

Nature of Dimensional Analysis

Example: Drag on a Sphere

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

- ✓ **Only one dependent and one independent variable**
- ✓ **Easy to set up experiments to determine dependency**
- ✓ **Easy to present results (one graph)**

BUCKINGHAM PI THEOREM

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The scheme given here was proposed in 1914 by Buckingham [29] and is now called the *Buckingham pi theorem*. The name *pi* comes from the mathematical notation Π, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by Π_1, Π_2, Π_3, etc. The method allows the pis to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

> If a physical process satisfies the PDH and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or Π's. The reduction $j = n - k$ equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables F, L, U, ρ, and μ described by three dimensions $\{MLT\}$. Thus $n = 5$ and $j \leq 3$. Therefore it is a good guess that we can reduce the problem to k pis, with $k = n - j \geq 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pis than this minimum (see Example 5.5)

BUCKINGHAM PI THEOREM

The second part of the theorem shows how to find the pis one at a time:

Find the reduction j, then select j scaling variables which do not form a pi among themselves.[4] Each desired pi group will be a power product of these j variables plus one additional variable which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose that the process involves five variables

$$v_1 = f(v_2, v_3, v_4, v_5)$$

Suppose that there are three dimensions $\{MLT\}$ and we search around and find that indeed $j = 3$. Then $k = 5 - 3 = 2$ and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables which do *not* form a pi, and suppose these turn out to be v_2, v_3, and v_4. Then the two pi groups are formed by power products of these three plus one additional variable, either v_1 or v_5:

$$\Pi_1 = (v_2)^a(v_3)^b(v_4)^c v_1 = M^0L^0T^0 \qquad \Pi_2 = (v_2)^a(v_3)^b(v_4)^c v_5 = M^0L^0T^0$$

Here we have arbitrarily chosen v_1 and v_5, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of a, b, and c for each pi. And they are independent because only Π_1

BUCKINGHAM PI THEOREM

- **Step 1:**

  List all the parameters involved

  Let ***n*** be the number of parameters

  Example: For drag on a sphere, F, V, D, ρ, μ, & $n = 5$

- **Step 2:**

  Select a set of primary dimensions

  For example *M (kg), L (m), t (sec).*

  Example: For drag on a sphere choose *MLt*

BUCKINGHAM PI THEOREM

- **Step 3**

  List the dimensions of all parameters

  Let r be the number of primary dimensions

  Example: For drag on a sphere $\boldsymbol{r = 3}$

F	V	D	ρ	μ
$\frac{ML}{t^2}$	$\frac{L}{t}$	L	$\frac{M}{L^3}$	$\frac{M}{Lt}$

Buckingham Pi Theorem

- **Step 4**

  Select a set of ***r*** *dimensional parameters that includes all the primary dimensions*

  Example: For drag on a sphere ($m = r = 3$) select ρ, V, D

Buckingham Pi Theorem

- Step 5

  Set up dimensionless groups π^s

  There will be $n - m$ equations

  Example: For drag on a sphere

$$\Pi_1 = \rho^a V^b D^c F$$

$$\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{ML}{t^2}\right) = M^0 L^0 t^0$$

$$\Pi_1 = \frac{F}{\rho V^2 D^2}$$

Buckingham Pi Theorem

- **Step 6**

  Check to see that each group obtained is dimensionless

  Example: For drag on a sphere

$$[\Pi_1] = \left[\frac{F}{\rho V^2 D^2}\right]$$

$$F\frac{L^4}{Ft^2}\left(\frac{t}{L}\right)^2\frac{1}{L^2} = 1$$

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

Length: $$a + b - 3c + 1 = 0$$

Mass: $$c - 1 = 0$$

Time: $$-b \qquad + 1 = 0$$

from which we find

$$a = b = c = 1$$

Therefore $$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \text{Re}$$ *Ans.*

We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$ *Ans.*

which is exactly Eq. (5.2).

Significant Dimensionless Groups in Fluid Mechanics

- Reynolds Number

$$Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

✓ **Mach Number**

$$M = \frac{V}{c}$$

Significant Dimensionless Groups in Fluid Mechanics

- Froude Number

$$Fr = \frac{V}{\sqrt{gL}}$$

✓ **Weber Number**

$$We = \frac{\rho V^2 L}{\sigma}$$

Significant Dimensionless Groups in Fluid Mechanics

- Euler Number

$$Eu = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

✓ **Cavitation Number**

$$Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$$