Dimensional Analysis and Similitude

Modeling at aeromechanical experiment. Similarity criterion. (Re, Pe, Fr, Nu, Pr, Sh and etc.)

Main Topics

- Nature of Dimensional Analysis
- Buckingham Pi Theorem
- Significant Dimensionless Groups in Fluid Mechanics
- Flow Similarity and Model Studies

Objectives

- 1. Understand dimensions, units, and dimensional homogeneity
- 2. Understand benefits of dimensional analysis
- 3. Know how to use the method of repeating variables
- 4. Understand the concept of similarity and how to apply it to experimental modeling

Dimensions and Units

Review

- Keview
- Dimension: Measure of a physical quantity, e.g., length, time, mass
- <u>Units:</u> Assignment of a number to a dimension, e.g., (m), (sec), (kg)

7 Primary Dimensions:

1.	Mass	m	(kg)
2.	Length	L	(m)
3.	Time	t	(sec)
4.	Temperature	т	(K)
5.	Current	1.1	(A)
6.	Amount of Light	С	(cd)
7.	Amount of matter	Ν	(mol)

Dimensions and Units

- All non-primary dimensions can be formed by a combination of the 7 primary dimensions
- Examples
 - {Velocity} m/sec = {Length/Time} = {L/t}
 - Force} N = {Mass Length/Time} = {mL/t²}

Dimensional Homogeneity

- Every additive term in an equation must have the same dimensions
- Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho gz = C$$

- > {p} = {force/area}={mass x length/time x 1/length²} = {m/(t²L)}
- ► ${1/2\rho V^2} = {mass/length^3 x (length/time)^2} = {m/(t^2L)}$
- \ {\rhogz} = {mass/length³ x length/time² x length}
 ={m/(t²L)}

Nature of Dimensional Analysis

Example: Drag on a Sphere

$$F = f(D, V, \rho, \mu)$$

- Drag depends on FOUR parameters: sphere size (D); speed (V); fluid density (ρ); fluid viscosity (μ)
- Difficult to know how to set up experiments to determine dependencies
- Difficult to know how to present results (four graphs?)

Nature of Dimensional Analysis

Example: Drag on a Sphere

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

- Only one dependent and one independent variable
- Easy to set up experiments to determine dependency
- Easy to present results (one graph)

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The scheme given here was proposed in 1914 by Buckingham [29] and is now called the *Buckingham pi theorem*. The name *pi* comes from the mathematical notation Π , meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by Π_1 , Π_2 , Π_3 , etc. The method allows the pis to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

If a physical process satisfies the PDH and involves *n* dimensional variables, it can be reduced to a relation between only *k* dimensionless variables or Π 's. The reduction j = n - k equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables F, L, U, ρ , and μ described by three dimensions {*MLT*}. Thus n = 5 and $j \le 3$. Therefore it is a good guess that we can reduce the problem to k pis, with $k = n - j \ge 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pis than this minimum (see Example 5.5)

The second part of the theorem shows how to find the pis one at a time:

Find the reduction j, then select j scaling variables which do not form a pi among themselves.⁴ Each desired pi group will be a power product of these j variables plus one additional variable which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose that the process involves five variables

$$v_1 = f(v_2, v_3, v_4, v_5)$$

Suppose that there are three dimensions {*MLT*} and we search around and find that indeed j = 3. Then k = 5 - 3 = 2 and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables which do *not* form a pi, and suppose these turn out to be v_2 , v_3 , and v_4 . Then the two pi groups are formed by power products of these three plus one additional variable, either v_1 or v_5 :

$$\Pi_1 = (v_2)^a (v_3)^b (v_4)^c v_1 = M^0 L^0 T^0 \qquad \Pi_2 = (v_2)^a (v_3)^b (v_4)^c v_5 = M^0 L^0 T^0$$

Here we have arbitrarily chosen v_1 and v_5 , the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of *a*, *b*, and *c* for each pi. And they are independent because only Π_1

Step 1:

List all the parameters involved Let *n* be the number of parameters Example: For drag on a sphere, *F*, *V*, *D*, ρ , μ , & n = 5

Step 2:

Select a set of primary dimensions

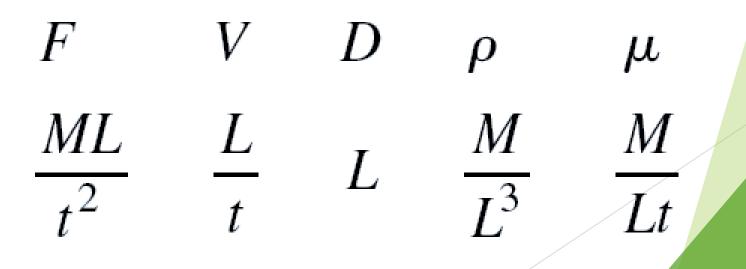
For example M(kg), L(m), t(sec).

Example: For drag on a sphere choose *MLt*

Step 3

List the dimensions of all parameters Let *r* be the number of primary dimensions

Example: For drag on a sphere r = 3



Buckingham Pi Theorem



Select a set of **r** dimensional parameters that includes all the primary dimensions

Example: For drag on a sphere (m = r = 3)select ϱ , V, D

Buckingham Pi Theorem

► Step 5

Set up dimensionless groups π^s There will be n - m equations Example: For drag on a sphere

$$\begin{split} \Pi_1 &= \rho^a V^b D^c F\\ \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{ML}{t^2}\right) &= M^0 L^0 t^0\\ \Pi_1 &= \frac{F}{\rho V^2 D^2} \end{split}$$

Buckingham Pi Theorem

Step 6

Check to see that each group obtained is dimensionless Example: For drag on a sphere

$$[\Pi_1] = \left[\frac{F}{\rho V^2 D^2}\right]$$

$$F\frac{L^4}{Ft^2}\left(\frac{t}{L}\right)^2\frac{1}{L^2} = 1$$

Finally, add viscosity to L, U, and ρ to find Π_2 . Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

Length:	a+b-3c+1=0		
Mass:	c - 1 = 0		
Time:	-b + 1 = 0		
from which we find			
	a = b = c = 1		

Therefore

$$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \text{Re}$$
 Ans.

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We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho UL}{\mu}\right)$$
 Ans

which is exactly Eq. (5.2).

Significant Dimensionless **Groups in Fluid Mechanics** $Re = \frac{\rho VL}{\mu}$ **Reynolds Number** $M = \frac{V}{V}$ Mach Number

Significant Dimensionless Groups in Fluid Mechanics

Froude Number

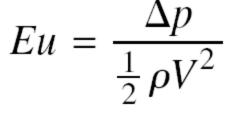
Weber Number

 $We = \frac{\rho V^2 L}{2}$

 $Fr = \frac{v}{\sqrt{gL}}$

Significant Dimensionless Groups in Fluid Mechanics

Euler Number



✓ Cavitation Number (

$$Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$$